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THESIS

TWO-PERSON ZERO-SUM NETWORK-INTERDICTION GAME WITH MULTIPLE INSPECTOR TYPES

by

Omur Unsal

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Thesis Advisor:	R. Kevin Wood
Second Reader:	Javier Salmeron

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This thesis extends the game-theoretic network-interdiction model of Washburn & Wood (1995) to handle multiple types of interdiction assets (e.g., aircraft, ground-based inspection teams), referred to here as "inspectors." A single evader attempts to traverse a path between two vertices in a directed network while an interdictor, controlling inspectors of different types, attempts to detect the evader by assigning inspectors to edges in the network. Each edge has a known probability of detection if the evader traverses the edge when an inspector of a given type is present. The problem for the interdictor is to find a mixed inspector-to-edge assignment strategy that maximizes the average probability of detecting the evader, i.e., the "interdiction probability." The problem for the evader is to find a mixed "path-selection strategy" that minimizes the interdiction probability. The problem is formulated as a two-person zero-sum game with a surrogate objective that evaluates expected number of detections. That model is solved with a "direct solution procedure" and a "marginal-probability solution procedure." On numerous test problems, both procedures correctly compute expected number of detections, but the latter more often finds a solution that simultaneously optimizes interdiction probability. The latter procedure is also much faster and is therefore preferred.

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TWO-PERSON ZERO-SUM NETWORK-INTERDICTION GAME WITH MULTIPLE INSPECTOR TYPES

Omur Unsal
Captain, Turkish Army
B.S., Turkish Land Force Academy, 2001

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Author: Omur Unsal

Approved by: Dr. R. Kevin Wood
Thesis Advisor

Dr. Javier Salmeron
Second Reader

Dr. Robert Dell
Chairman, Department of Operations Research

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This thesis extends the game-theoretic network-interdiction model of Washburn & Wood (1995) to handle multiple types of interdiction assets (e.g., aircraft, ground-based inspection teams), referred to here as "inspectors." A single evader attempts to traverse a path between two vertices in a directed network while an interdictor, controlling inspectors of different types, attempts to detect the evader by assigning inspectors to edges in the network. Each edge has a known probability of detection if the evader traverses the edge when an inspector of a given type is present. The problem for the interdictor is to find a mixed inspector-to-edge assignment strategy that maximizes the average probability of detecting the evader, i.e., the "interdiction probability." The problem for the evader is to find a mixed "path-selection strategy" that minimizes the interdiction probability.

The problem is formulated as a two-person zero-sum game with a surrogate objective that evaluates expected number of detections. That model is solved with a "direct solution procedure" and a "marginal-probability solution procedure." On numerous test problems, both procedures correctly compute expected number of detections, but the latter more often finds a solution that simultaneously optimizes interdiction probability. The latter procedure is also much faster and is therefore preferred.

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EXECUTIVE SUMMARY

This thesis extends the game-theoretic network-interdiction model of Washburn & Wood (1995) to handle multiple types of interdiction assets (e.g., aircraft, ground-based inspection teams), referred to here as "inspectors." A single evader attempts to traverse a path between two vertices in a directed network while an interdictor, controlling inspectors of different types, attempts to detect the evader by assigning inspectors to individual edges in the network. Each edge has a known probability of detection if the evader traverses the edge when an inspector of a given type is present. The problem for the interdictor is to find a mixed inspector-to-edge assignment strategy that maximizes the average probability of detecting the evader, i.e., the "interdiction probability." The problem for the evader is to find a mixed "path-selection strategy" that minimizes the interdiction probability.

We use two procedures to find an optimal solution to the network-interdiction problem described above. The first is the "direct solution procedure," which uses a column generation algorithm to find the pure inspector assignment strategy in a single phase. The second one is a "marginal-probability (solution) procedure" that (a) first solves a relaxed model to identify what is hoped to be an optimal marginal probability distribution for inspector-to-edge assignments, and then (b) uses a column-generation algorithm to find a mixed strategy, i.e., a joint distribution, that matches the marginal distribution, if this is possible.

The problem is formulated as a two-person zero-sum game with a surrogate objective that evaluates expected number of detections. Ideally, we would like that objective to measure probability of detection, i.e., "interdiction probability." When the number of inspectors does not exceed the cardinality of a minimum-cardinality cut in the network, both models optimize expected number of detections correctly. However, that is not true for the interdiction probability in the network. The marginal-probability procedure often finds a mixed strategy in which probability of detection equals expected number of detections, while the direct solution procedure does not, but the opposite never happens.

Multiple tests of both procedures are made to (a) find optimal inspection strategies, (b) to identify cases, if any, that the procedures do not solve correctly, and (c) to specify conditions under which the procedures are guaranteed to work or not work.

On numerous test problems, both procedures correctly compute expected number of detections, but the marginal-probability procedure more often finds a solution that simultaneously optimizes interdiction probability. In particular, the direct solution procedure sometimes places inspectors on two or more edges that appear in single path that the evader will use with positive probability; but probability of detection is not additive in this case. The marginal-probability procedure, on the other hand, always returns the correct answer given that the number of inspectors is less than the cardinality of minimum cardinality cut in the network. (Strictly speaking, we mean minimum-cardinality among cuts that contain no edges with a zero probability of detection.) This is true because, under these conditions, this procedure can never place more than a single inspector on a path that the evader might use.

The marginal-probability procedure also has an advantage in terms of solution times. For example, the directed-solution procedure solves one problem with 145 vertices, 544 edges and 12 inspectors (six of each of two types) in about half an hour on a fast personal computer; in contrast, the marginal probability procedure solves the same problem in about 15 seconds. Because of the advantages in accuracy and speed, the marginal-probability procedure is clearly preferred.

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I. INTRODUCTION

This thesis models and solves a two-person, zero-sum, network-interdiction game. The model represents an “interdictor” who wishes to assign “inspectors” to cover edges (links) of a network, in an attempt to detect an “evader,” moving through that network between two specified vertices. Imperfect detections and a limited number of inspectors lead to a simultaneous-play (Cournot) game in which the interdictor chooses a probabilistic inspector-to-edge assignment strategy, while the evader chooses a probabilistic “path-selection strategy.” The interdictor seeks to maximize the probability of detection and the evader to minimize that probability, although we model probability of detection through a surrogate, expected number of detections.

The model applies to the problem of optimally allocating search assets of a border-control authority seeking to detect illegal border crossings.

A. BACKGROUND

This thesis extends the game-theoretic network-interdiction model described by Washburn & Wood (1995) to handle multiple types of interdiction assets, referred to here as *inspectors*. The Washburn and Wood model represents a situation in which a single *evader* attempts to traverse a path between two vertices in a network while a single inspector tries to detect the evader by setting up an inspection point along one of the network edges. The inspector selects an edge e in the network and sets up an inspection site there. Without knowledge of the inspector’s location, the evader chooses a path. If the evader traverses edge e on that path, he is detected with probability p_e; otherwise, he goes undetected. Both the evader and inspector know the detection probability for each edge in the network. The problem for the inspector is to find a probabilistic “edge-inspection strategy” that maximizes the overall probability of detecting the evader—we call this the *interdiction probability*—while the problem for the evader is to find a probabilistic “path-selection strategy” that minimizes the interdiction probability. Washburn and Wood formulate this problem as a two-person zero-sum matrix game, and

then show how it can be solved using a maximum-flow network model. They also show how the solution extends to multiple, homogeneous inspectors, under certain circumstances.

The thesis extends Washburn and Wood model by modeling an *interdictor* who controls multiple types of inspectors. The interdictor has m_r inspectors available of each type $r \in R$. For instance if $R = \{1, 2\}$, $r = 1$ might represent a single type of UAV (unmanned aerial vehicle), and $r = 2$ might represent "inspection teams" that consist of ground forces. In this thesis we assume that the inspectors detect independently of each other, and that the number of inspectors per edge is limited to one. Although not critical to the formulation, that the reader may assume that $|R|$ is small, say three or four. (For instance, the interdictor might control two types of UAVs, one type of manned aircraft and a group of homogeneous ground teams, implying $|R| = 4$.) The evader's pure strategies correspond to paths as in Washburn and Wood model, but the interdictor's pure strategies, called the "inspector-assignment strategies," are more complex in the new model.

B. LITERATURE REVIEW

The literature on network interdiction can be divided roughly into two categories: sequential-play games (Stackelberg games) and simultaneous-play games (Cournot games). We are concerned with a simultaneous-play model, but briefly discuss sequential-play models here to clarify the differences. (See Gibbons 1997 for a general discussion and comparison of simultaneous-play and sequential-play games.)

In a sequential-play interdiction game, the interdictor and the evader play their strategies one after the other. For instance, consider a simplified version of the network-interdiction model described by Pan et al. (2003) (and which is related, using a logarithmic transformation in the objective function, to the "shortest-path interdiction models" of Fulkerson & Harding 1977, Golden 1978, and Israeli & Wood 2002). An interdictor controls multiple inspectors and plays first by placing those inspectors on edges in a network. An evader sees exactly where the inspectors have been placed and

chooses a path that minimizes some overall probability of detection; detection probability depends on probabilities of detection on individual edges and on which inspectors, if any, are encountered along the path. (There may be a non-zero “local probability of detection” on an edge even if an inspector is not present.) In other Stackelberg network-interdiction models, the interdictor may disrupt edges or vertices in the network so that the functionality that a “network user” can derive from the network is minimized. Functionality may be defined in such terms as maximum flow (Wood 1993), shortest-path length (Fulkerson & Harding 1977, Golden 1978, Israeli & Wood 2002), or cost of a multi-commodity flow (Lim & Smith 2007); see also Wollmer (1964). Bayrak & Bailey (2008) consider a more complicated model involving asymmetric information.

This thesis is concerned with the second main type of network-interdiction model, however, the type involving simultaneous play. In a simultaneous-play network-interdiction game, the interdictor and evader act at the same time, or at least without knowing what each other’s strategy is.

Washburn & Wood (1995) describe what appears to be the first simultaneous-play game that is called a “network-interdiction game,” but at least one earlier simultaneous-play model has the flavor of network interdiction. In particular, Caulkins et al. (1993) investigate how the adaptive reactions of an evader to an interdictor’s strategies affect interdiction probability in a network. The authors use Monte Carlo simulation for an interdiction game involving an interdictor and cocaine smugglers. Basically, the smugglers make shipments on “routes,” and the interdictor tries to detect those shipments by searching some of the routes. A smuggler computes the risk of being interdicted by using a “time-weighted estimate” of risk for each route, which is based on the interdiction history of the routes; he chooses a route for the a shipment probabilistically, using probabilities that are inversely proportional to the risk estimates. We note that the network of routes in this model is extremely simple and, in fact, may be viewed as a set of parallel edges between a single origin and single destination.

We find numerous references to Washburn & Wood (1995) in the literature, but find only one paper that studies a model having a close connection to that of Washburn and Wood, viz., Kodialam & Lakshman (2003). Kodialam and Lakshman model a game-

theoretic situation in which an “intruder” attempts to inject a malicious package into a vertex of communications network, and would like the package to reach a given target vertex undetected. Simultaneously, a “service provider” inspects a subset of the packages as the move across the network’s links, in an attempt to detect the intrusion. That is, the service provider samples the flow of packages on certain network edges. Sampling must be used because it is computationally impossible to check all packages. The intruder tries to hide his package by sending it along edges that have high nominal levels of package flow. Detection probability on an edge corresponds to the proportion of package flow that is sampled. This model differs from other interdiction models because it uses sampling to detect “the intruder,” rather than using a discrete allocation of inspectors. As in Washburn and Wood (1995), the inspector’s (service provider’s) optimal mixed strategy involves a minimum-weight cut. In this case, and edge’s weight is inversely proportional to the magnitude of the nominal package flow on the edge.

C. THESIS OUTLINE

The thesis is outlined as follows. Chapter II introduces the two-person zero-sum network interdiction game with multiple inspector types; Chapter III describes a single-phase column-generation procedure, or “direct procedure” to solve the game; Chapter IV describes an alternative, two-phase “marginal-probability procedure”; Chapter V presents basic computational results for three test problems with different networks and with up to three inspector types; and Chapter VI presents summary, conclusions, and suggestions for further research.

II. MODELING TWO-PERSON ZERO-SUM NETWORK INTERDICTION GAME WITH MULTIPLE INSPECTOR TYPES

The key limitation of the model of Washburn & Wood (1995) is that of a single inspector type. That assumption might be appropriate if the interdictor controls only, say, ground-based, mobile inspection teams, but is unrealistic if he “inspects” the road network with several types of assets such as (a) unmanned aerial vehicles (UAVs) of a particular type, (b) UAVs of a different type, (c) electronic sensors, and (d) ground-based inspection teams. Each type of inspector will normally have a different probability of detection, which is likely to depend on location. For instance, a UAV might have a relatively high probability of detecting an evader in mountainous terrain, while a ground team might be less effective over the same area. In contrast, a ground team might be a better choice over UAV in an area in which harsh weather conditions prevail. Thus, our key goal is to extend Washburn and Wood model to handle multiple inspector types.

In our model, we wish to maximize the interdictor’s overall probability of detection, i.e., the interdiction probability, but will write an objective function that computes expected number of detections because:

1. That objective is easy to define and simplifies modeling,
2. It corresponds closely to probability of detection when overall detection probabilities are small−this is a realistic situation−and,
3. It corresponds exactly to probability of detection if an optimal inspection policy means that the evader has only one detection opportunity on any path he might choose with positive probability.

Since inspection assets are likely to be limited compared to the size of the network, the situation under point three is likely to occur.

This chapter begins by defining the notation that will be used throughout the thesis, and starts to discuss how the extension from Washburn and Wood’s single-inspector-type model to our multiple-inspector-type model will be achieved. It then discusses the structure of a matrix game for a general network-interdiction model, and finally describes the extended model.

A. NOTATION

The following notation will be used throughout this thesis:

Indices and Index Sets

$i, j \in V$ vertices in a directed network $G = (V, E)$

s, t special vertices $s \neq t \in V$. s is the "source vertex" where the evader will start his traversal through the network, and t is the "sink vertex" which is the evader's final destination (after reaching t, the evader cannot be detected)

$e \in E$ edges $e = (i, j)$ and $e = (j, i)$ in a directed network $G = (V, E)$

$e \in E'$ edges, incident from source vertex (s,v) and to sink vertex (v,t), which have zero probabilities; $E' \subset E$

$\ell \in L$ s-t paths in $G = (V, E)$ (pure strategies for the evader)

$r \in R$ inspector types

$a \in A$ a pure inspection strategy, i.e., an assignment of inspectors to edges in G

Data

$g_{\ell e}$ 1 if s-t path ℓ contains edge e, and 0 otherwise

h_{era} 1 if inspection strategy a puts an inspector of type r on the edge e, and 0 otherwise

p_{er} probability an inspector of type r, when assigned to edge e, will detect the evader if the evader traverses that edge

$\hat{d}_{a\ell}$ expected number of detections achieved by pure inspection strategy a if the evader traverses path ℓ ($\hat{d}_{a\ell} = \sum_{e \in E} \sum_{r \in R} g_{\ell e} h_{era} p_{er}$)

m_r number of inspectors of type r available for assignment

F vertex-edge incidence matrix for G; $f_{ie} = 1$ if $e = (i, j) \in E$, $f_{ie} = -1$ if $e = (j, i) \in E$, and $f_{ie} = 0$ otherwise

b_i $b_s = 1, b_t = -1$ and $b_i = 0 \ \forall i \in V \setminus \{s, t\}$)

Variables

For the "basic model":

x'_a probability that interdictor chooses pure edge-inspection strategy a (vector form $\mathbf{x}'$)

y'_ℓ probability that the evader chooses path ℓ (vector form $\mathbf{y}'$)

y_e (marginal) probability that the evader traverses edge e ($y_e = \sum_{\ell \in L} g_{\ell e} y'_\ell$; vector form $\mathbf{y}$)

For the "marginal-probability procedure":

x_{er} (marginal) probability that an inspector of type r is assigned to edge e (vector form $\mathbf{x}$)

v_{er} unassigned probability which we are trying to minimize

For the "column-generation procedure" used after computing marginal probabilities:

h_{er} 1 if an inspector of type r is assigned to edge e, and 0 otherwise

B. A MATRIX GAME FOR NETWORK INTERDICTION MODEL

The basic model in this thesis has the same two players as in the Washburn and Wood model, but the interdictor will control m_r inspectors of each type $r \in R$. The pure strategies for the evader are the same as in Washburn and Wood model, i.e., they will correspond to paths. A pure strategy for the interdictor involves a generalization, however: simultaneous assignments of multiple inspectors to edges. A pure strategy, indexed by a, is called an *inspector-assignment strategy*, or simply an *inspection strategy*. The interdictor will normally have many potential, pure inspection strategies, and each such strategy will have a probability of being used by the interdictor: together, those define a *mixed inspector-assignment strategy*, or simply a *mixed assignment strategy*. For simplicity, we allow at most one inspector on any edge in any pure strategy. Also for simplicity, we assume that an inspector of any type when assigned to edge (i, j) can detect an evader who travels over that edge, but cannot detect one who travels over the anti-parallel edge (j, i).

Let x'_a denote the probability that the interdictor chooses inspection strategy a for the multiple inspectors under his control, and let y'_ℓ denote the path-selection

probability for the evader, which is the probability that the evader chooses path ℓ to traverse. As in Washburn & Wood (1995), the interdictor seeks a randomized strategy $\mathbf{x}'$ that maximizes the expected number of detections of the evader, given that the evader uses a randomized strategy $\mathbf{y}'$ that minimizes the same objective. Let $\hat{d}_{a\ell}$ be the expected number of detections if evader chooses path ℓ and the interdictor chooses inspection strategy a. This problem fits into the form of a matrix game with matrix $\hat{D} = [\hat{d}_{a\ell}]$ and can be stated as follows.

MAXMIN1

$$z_{\text{MAXMIN1}} = \max_{\mathbf{x}' \geq \mathbf{0}} \min_{\mathbf{y}' \geq \mathbf{0}} \mathbf{x}' \hat{D} \mathbf{y}' \tag{1}$$

$$\text{s.t.} \sum_{a \in A} x'_a = 1 \tag{2}$$

$$\sum_{\ell \in L} y'_\ell = 1 \tag{3}$$

The objective function (1) evaluates expected number of detections. Constraint (2) requires that the interdictor choose a valid mixed strategy of assignments; constraint (3) requires that the evader choose a valid mixed path-selection strategy.

Under certain circumstances, the expected number of detections, z_{MAXMIN1} or "E_D," equals the interdiction probability, which will be denoted P_I. This is possible when: (a) the interdictor's inspector-assignment strategies put the inspectors on edges that form an *s*-*t* cut, i.e., a disconnecting set, (b) the pure strategies do not place the inspectors on subsequent edges, or do not assign more than one inspectors on the edges of the same path ℓ, and (c) none of the pure strategies puts more than one inspector to any individual edge.

We can reformulate **MAXMIN1** as:

MAXMIN2

$$z_{\text{MAXMIN2}} = \max_{\mathbf{x}'\geq\mathbf{0}} \min_{\mathbf{y}'\geq\mathbf{0}} \sum_{a\in A} x'_a \sum_{\ell\in L} \sum_{e\in E_\ell} \sum_{r\in R} g_{\ell e} h_{era} p_{er} y'_\ell \tag{4}$$

s.t. (2), (3)

where $h_{era}=1$ if pure inspector-assignment strategy a puts an inspector of type r on the edge e and $h_{era}=0$ otherwise; p_{er} is the probability that an inspector of type r detects the evader on e given that the evader traverses edge e; and x'_a is the probability that the interdictor chooses pure edge-inspection strategy a.

Defining y_e as the "edge-traversal probability" for the evader on edge e, we can write the new problem as follows:

MAXMIN3

$$z_{\text{MAXMIN3}} = \max_{\mathbf{x}'\geq\mathbf{0}} \min_{\mathbf{y}\geq\mathbf{0}} \sum_{a\in A} x'_a \sum_{e\in E} \left(\sum_{r\in R} h_{era} p_{er} \right) y_e \tag{5}$$

s.t. (2), (3),

$$y_e = \sum_{\ell\in L} g_{\ell e} y'_\ell \;\; \forall e\in E \tag{6}$$

$$F\mathbf{y} = \mathbf{b} \tag{7}$$

Washburn & Wood show that nothing is lost by eliminating constraints (3) and (6), and we follow that approach to arrive at the following model:

MAXMIN4

$$z_{\text{MAXMIN4}} = \max_{\mathbf{x}'\geq\mathbf{0}} \min_{\mathbf{y}\geq\mathbf{0}} \sum_{a\in A} x'_a \sum_{e\in E} \left(\sum_{r\in R} h_{era} p_{er} \right) y_e \qquad \text{[Dual Variables]} \tag{8}$$

$$\text{s.t.} \; \sum_{a\in A} x'_a = 1 \qquad [\theta] \tag{9}$$

$$F\mathbf{y} = \mathbf{b} \qquad [\gamma] \tag{10}$$

If we (a) fix the interdictor's probabilities of edge-inspection strategy $\mathbf{x}'$, (b) take dual of evader's resulting linear program and then, (c) release $\mathbf{x}'$, we obtain the following *master problem*:

MP1(A)

$$z_{\text{MP1}} = \max_{\mathbf{x}'\geq\mathbf{0},\,\boldsymbol{\gamma}\geq\mathbf{0}} \gamma_s - \gamma_t \qquad \text{[Dual Variables]} \qquad (11)$$

$$\text{s.t. } \gamma_i - \gamma_j - \sum_{a\in A}\left(\sum_{r\in R} h_{era} p_{er}\right) x'_a \leq 0 \;\; \forall e=(i,j)\in E \qquad [\pi_e] \qquad (12)$$

$$\sum_{a\in A} x'_a = 1 \qquad [\theta] \qquad (13)$$

The number of variables in **MP1**(A) will be exponential in $|E|$ and m_r, so the model cannot be solved directly for large networks. The next chapter describes a solution approach.

III. SOLVING THE MATRIX GAME BY COLUMN GENERATION

Setting up and solving **MP1**(A) directly will be impossible for networks of even modest size, because the number of variables or "columns" will be huge. But, because the number of constraints is modest it is possible to generate columns on an "as-needed" basis in a column-generation algorithm. We are going to use the "column generation technique" as described by Ahuja et al. (pp. 665–670):

> The key idea in column generation is never to list explicitly all of the columns of the problem formulation, but rather to generate them as needed. To find the simplex multipliers, the method requires no information about columns (variables) not in the basis. It then uses the multipliers to price-out the nonbasic columns, that is, compute their reduced costs. If any reduced cost is positive (assuming a maximization formulation), the method will introduce one nonbasic variable into the basis in place of one of the current basic variables, and then recompute the simplex multipliers, and repeat these computations.

The basics of this approach, when applied to solving the **MP1**(A), are similar to the column-generation algorithm for solving the multi-commodity flow problem (Ford & Fulkerson 1956). That algorithm is, in turn, a special case of a Dantzig-Wolfe decomposition algorithm for a general linear program (Dantzig & Wolfe 1958).

In this chapter, a subproblem will be defined for the master problem **MP1**(A) from the previous chapter, and an optimal mixed edge-inspection strategy will be produced by column generation. We will introduce a *single-phase column-generation procedure*, or simply *direct solution procedure* to find the interdictor's optimal mixed strategy.

A. COLUMN GENERATION SUBPROBLEM

Assume we have enumerated a subset of pure inspection strategies $A' \subset A$. We solve **MP1**(A') and retrieve optimal dual variables $(\hat{\boldsymbol{\pi}}, \hat{\theta})$. We may view that solution as a basic feasible solution for **MP1**(A). It may be possible to improve that solution if we can find a variable $x'_a, a \in A \setminus A'$ with positive reduced cost, i.e.,

$$0-\sum_{e\in E}\sum_{r\in R} p_{er}h_{era}\hat{\pi}_e-\hat{\theta}>0$$
.

The following *subproblem* generates a column (variable) with the most positive reduced cost, if one exists.

SUB1 $(\hat{\boldsymbol{\pi}},\hat{\theta})$

$$z_{\text{SUB1}}=\max_{\mathbf{h}}-\sum_{e\in E}\sum_{r\in R} p_{er}h_{er}\hat{\pi}_e-\hat{\theta} \tag{14}$$

s.t. $\sum_{e\in E} h_{er}=m_r\ \forall r\in R$ (15)

$$\sum_{r\in R} h_{er}\le 1\ \forall e\in E \tag{16}$$

$$h_{er}\in\{0,1\}\ \forall e\in E, r\in R \tag{17}$$

B. COLUMN GENERATION ALGORITHM TO SOLVE MP1(A)

To solve the **MP1**(A) problem efficiently, we will use the following column-generation algorithm.

Algorithm 1

Input: $G=(V,E)$; source vertex s; sink vertex t; $p_{er}\ \forall r\in R, e\in E$; $m_r\ \forall r\in R.$

/*Input assumes $\sum_r m_r \le |E|$.*/

Output: Optimal mixed strategy for **MP1**(A) encoded through $A'\subset A,\ h_{era}\ \forall e\in E, r\in R, a\in A'$ and probabilities $x'_a>0\forall a\in A'$.

1. **/*** Create an initial feasible assignment of inspectors. */
   $a\leftarrow 1;$
   for (all r in R) {for (i=1 to m_r) { e $\leftarrow$ e+1; $h_{era}\leftarrow 1;$ }}
   $A'\leftarrow\{a\};$
2. Solve **MP1** (A') for the dual solution $(\hat{\boldsymbol{\pi}},\hat{\theta})$ and primal solution $\hat{\mathbf{x}}'$;
3. Solve **SUB1** $(\hat{\boldsymbol{\pi}},\hat{\theta})$ for $(z_{sub},\hat{\mathbf{h}})$;
   If $(z_{sub}=0)$ {
   /*Current solution to **MP1** (A') solves **MP1** (A) */
   Print $(h_{era}\ \forall e\in E, r\in R, a\in A' \mid x'_a>0)$;
   Print $(x'_a\ \forall a\in A' \mid x'_a>0)$;
   Halt;

}else{

$a \leftarrow a+1$;
$h_{era} \leftarrow \hat{h}_{er} \; \forall e, r$;
$A' \leftarrow A' \cup \{a\}$;
go to step 2;
}

Suppose we wish to use **Algorithm 1** to solve two instances of **MP1**(A) on the network of Figure 1.

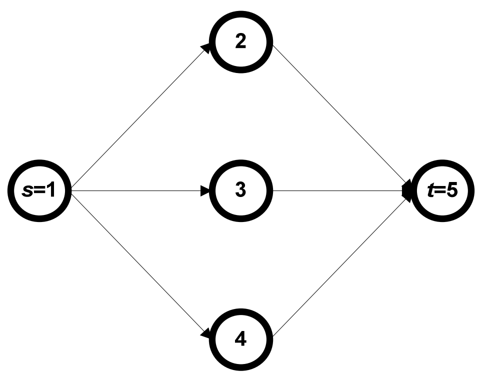

Figure 1. Small network with five vertices and six edges to illustrate a solution of the efficiency of direct solution procedure.

Figure 2 and Figure 3 use the network of Figure 1 to illustrate optimal mixed inspection strategies for an interdictor who controls one inspector (Figure 2), and two inspectors of a given type (Figure 3).

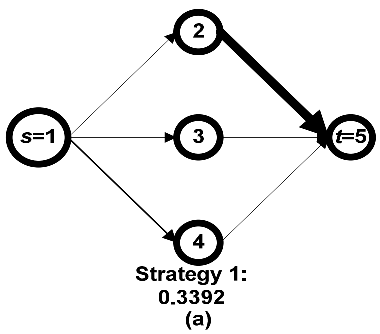

Figure 2. Optimal mixed assignment strategy for the interdictor. Results for direct solution procedure on the small network of Figure 1, when one inspector of type 1 is used only, and $p_{(s,2)1} = 0.35$, $p_{(s,3)1} = 0.56$, $p_{(s,4)1} = 0.51$, $p_{(2,5)1} = 0.52$, $p_{(3,5)1} = 0.45$, $p_{(4,5)1} = 0.38$. The solution gives $P_I = E_D = 0.1764$. The dark edges indicate the inspector assignment in each of three pure strategies, and the associated value for x'_a is given at the bottom of each subfigure. Note that the interdictor's inspector-to-edge assignments form an *s*-*t* cut in the network.

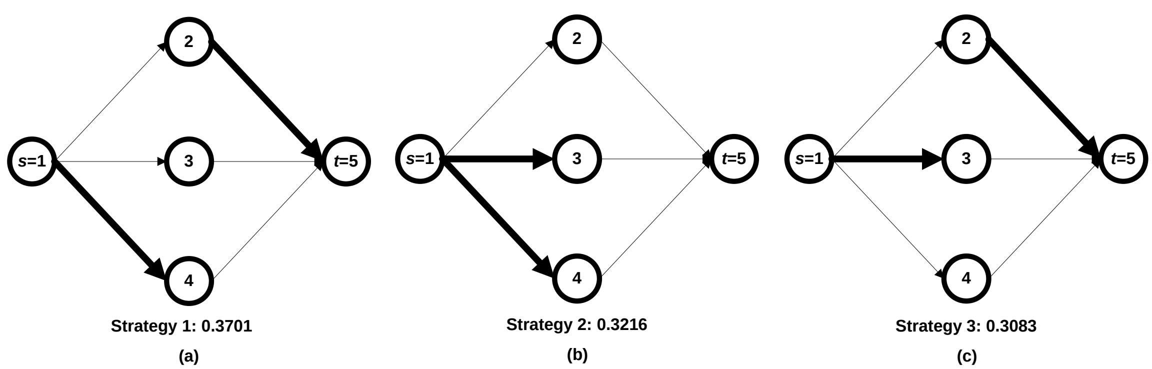

Figure 3. Optimal inspector assignments analogous to Figure 2 when two inspectors of type 1 are used only and no others, and $p_{(s,2)2} = 0.45$, $p_{(s,3)2} = 0.55$, $p_{(s,4)2} = 0.42$, $p_{(2,5)2} = 0.39$, $p_{(3,5)2} = 0.49$, and $p_{(4,5)2} = 0.52$. The solution gives $P_I = E_D = 0.3528$.

As can be seen in the first two examples, the direct solution procedure can produce acceptable strategies for the interdictor in which $P_I = E_D$. That is because the interdictor's pure inspector assignment strategies put at most one inspector on any path. However consider the previous example modified to have exactly two inspectors of type 1, and suppose all p_{e1} are the same, say $p_{e1} = 0.1, \forall e \in E$. Figure 4 displays the results:

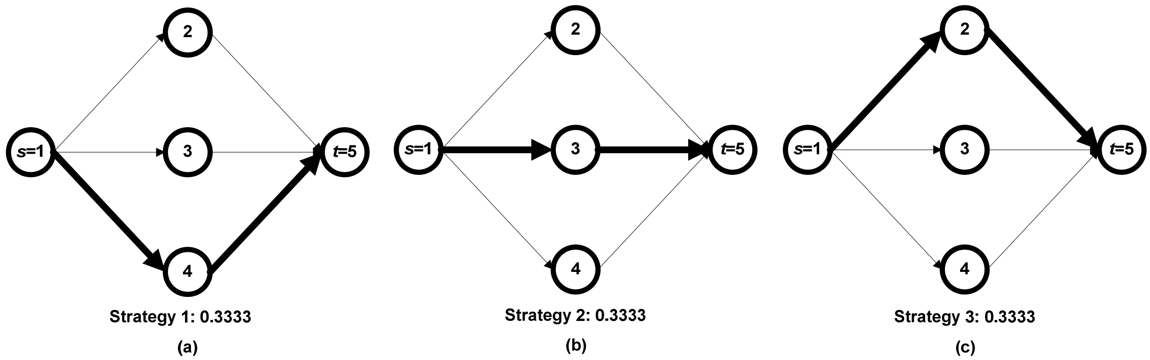

Figure 4. Optimal inspector assignments produced by the direct solution procedure when two inspectors of type 1 are used only (none from any other types), and $p_{e1} = 0.1 \, \forall e \in E$. The solution gives $E_D = 0.0667$. In this example, the edges with positive probability of inspection define an *s*-*t* cut but that cut is not minimal.

In this example, the direct solution procedure gives $E_D = 0.0667$. Each of the interdictor's pure strategies puts two inspectors on edges that appear in a single path, and thus $P_I \neq E_D$. In fact, $P_I = 0.0633$ in this solution. P_I and E_D do not differ by much here because the expected number of detections does not equal the actual interdiction probability here, because the p_{e1} are small. But, consider the same example but $p_{e1} = 0.9, \forall e \in E$. Then, the solution procedure yields $E_D = 0.60$, while $P_I = 0.33$. Actually there exists a solution with $P_I = E_D = 0.60$ (for example, the mixed inspection strategy that chooses each pair of edges incident to *s* with equal probability), but the direct solution procedure does not find it. Thus, we can conclude that the model is potentially useful and sometimes produces correct solutions, but can also give incorrect solutions. To circumvent this difficulty, at least part of the time, the next chapter describes a solution procedure based on marginal probabilities.

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IV. MARGINAL-PROBABILITY PROCEDURE

The solution procedure for **MAXMIN3** described in this chapter first finds a set of "marginal probabilities" for inspector assignments that satisfies a relaxed version of that model. Those marginal probabilities represent how much the edges should be used in the interdictor's mixed strategy, if possible. Then, column generation is used in an attempt to find a mixed inspector-assignment strategy that matches those marginal probabilities. If a valid mixed strategy is found, it must be optimal. This approach has two advantages over solution of the direct solution procedure: (a) if the number of inspectors does not exceed the cardinality of a minimum cardinality cut, by construction, no strategy will ever be found in which an evader, acting optimally, will encounter more than one inspector, so the objective function does represent interdiction probability, and (b) computational efficiency can be much better. Since this new solution procedure first finds the marginal probabilities for each edge, and then by using those marginal probabilities tries to find a mixed inspector-assignment strategy for the interdictor; we will call the procedure *two-phase marginal-probability and column-generation procedure*, or simply *marginal-probability procedure*.

A. A MARGINAL-PROBABILITY MODEL FOR MAXMIN3

The following model, **MAXMIN4**, expresses **MAXMIN3** in terms of marginal edge-inspection probabilities x_{er}. **MAXMIN4** is a relaxation of **MAXMIN3**, but if a valid mixed strategy can then be found that matches the marginal probability distribution, that distribution and the corresponding mixed strategy must be optimal. The new model is:

MAXMIN4

$$z_{\mathrm{MAXMIN4}} = \max_{\mathbf{x}\geq\mathbf{0}} \min_{\mathbf{y}\geq\mathbf{0}} \sum_{e\in E}\left(\sum_{r\in R} x_{er}\, p_{er}\right) y_e \qquad \text{[dual variables]} \qquad (18)$$

$$\text{s.t. } \sum_{e\in E} x_{er} \leq m_r \ \forall r \in R \qquad [\alpha_r] \qquad (19)$$

$$\sum_{r\in R} x_{er} \leq 1\, \forall e \in E \qquad [\beta_e] \qquad (20)$$

$$F\mathbf{y} = \mathbf{b} \qquad [\boldsymbol{\gamma}] \qquad (21)$$

The objective function (18) evaluates the probability of detection of the evader. Constraints (19) require that, on average, all inspectors be assigned to an edge; constraints (20) require that, on average, no more than one inspector may be assigned to any edge. Flow-balance constraints for the evader, (21), represent a relaxed version of the constraints that require the path-traversal probabilities sum to one.

Now let us move from **MAXMIN4** to complete the procedure using column generation. To solve this problem for the evader's optimal strategy, we will (a) fix evader's edge traversal probabilities **y**, (b) take dual of interdictor's resulting linear program and then, (c) release **y**. The following model results:

MIN4

$$z_{\mathrm{MIN4}} = \min_{\mathbf{y}\geq\mathbf{0},\boldsymbol{\alpha}\geq\mathbf{0},\boldsymbol{\beta}\geq\mathbf{0}} \sum_{r\in R} \alpha_r m_r + \sum_{e\in E} \beta_e \qquad (22)$$

$$\text{s.t. } \alpha_r + \beta_e - p_{er} y_e \geq 0 \ \forall e \in E, r \in R \qquad (23)$$

$$F\mathbf{y} = \mathbf{b} \qquad (24)$$

Letting $\boldsymbol{\gamma}$ denote the vector of dual variables for the flow-balance constraint (21), we can create a model to compute optimal marginal, inspector-assignment probabilities x_{er}:

MAX4

$$z_{\text{MAX4}} = \max_{\mathbf{x}\geq\mathbf{0},\boldsymbol{\gamma}} \gamma_s - \gamma_t \tag{25}$$

$$\text{s.t. } \gamma_i - \gamma_j - \sum_{r\in R} p_{er} x_{er} \leq 0 \;\forall e = (i,j) \in E \tag{26}$$

$$\sum_{e\in E} x_{er} \leq m_r \;\forall r \in R \tag{27}$$

$$\sum_{r\in R} x_{er} \leq 1 \;\forall e \in E \tag{28}$$

A new "probability-allocation model" uses column generation to find a subset of assignments $A' \subset A$, and associated probabilities, to match the marginal probabilities computed in **MAX4** if this is possible:

MP2 (A)

$$z_{\mathbf{MP2}} = \min_{\mathbf{v}\geq\mathbf{0},\mathbf{x}'\geq\mathbf{0}} \sum_{e\in E}\sum_{r\in R} v_{er} \qquad \text{[dual variables]} \tag{29}$$

$$\text{s.t. } v_{er} + \sum_{a\in A'} h_{era} x'_a = \hat{x}_{er} \;\forall e \in E, r \in R \qquad [\pi_{er}] \tag{30}$$

$$\sum_{a\in A'} x'_a = 1, \qquad [\theta] \tag{31}$$

where $h_{era} = 1$ if strategy a puts an inspector of type r on the edge e, and $h_{era} = 0$ otherwise; v_{er} is "unallocated probability," the sum of which we are trying to minimize; and $\hat{x}_{er}$ is the probability that an inspector of type r is assigned on edge e (the marginal probabilities obtained from **MAX4**.)

We will try to generate a "column" of **MP2** (A) that has negative reduced cost. Any column corresponds to a pure assignment strategy for the interdictor, and thus the following column-generation subproblem results:

SUB2 $(\hat{\boldsymbol{\pi}}, \hat{\theta})$

$$z_{\text{SUB2}} = \min_{\mathbf{h}} -\sum_{e\in E}\sum_{r\in R} \hat{\pi}_{er} h_{er} - \hat{\theta} \tag{32}$$

$$\text{s.t.} \sum_{r\in R} h_{er} \le 1 \; \forall e \in E \tag{33}$$

$$\sum_{e\in E} h_{er} = m_r \;\; \forall r \in R \tag{34}$$

$$h_{er} \in \{0,1\} \; \forall e, r \tag{35}$$

Given a solution $\hat{h}_{er}$ to **SUB2** $(\hat{\boldsymbol{\pi}}, \hat{\theta})$ for all $e \in E$ and $r \in R$, we define a new column index a', let $h_{era'} = \hat{h}_{er} \;\forall e \in E, r \in R$, add a' to A', and return to **MP2** (A')**.** This procedure continues until no new columns with negative reduced cost can be found, or until $v_{er} = 0 \;\forall e, r$. If all $v_{er} = 0,$ we have solved the original model **MAXMIN1**.

Now, let us attempt to overcome the difficulty we encountered in the direct solution procedure in Chapter III by using the marginal-probability procedure. The interdictor has two inspectors of type 1 and none from any other types, and $p_{e1} = 0.1$ for all edges e. The marginal-probability procedure produces the results shown in Figure 5, for which $E_D = P_I = 0.0667.$ The marginal probabilities produced by **MAX4**, in the first phase of the solution procedure are all 0.3333.

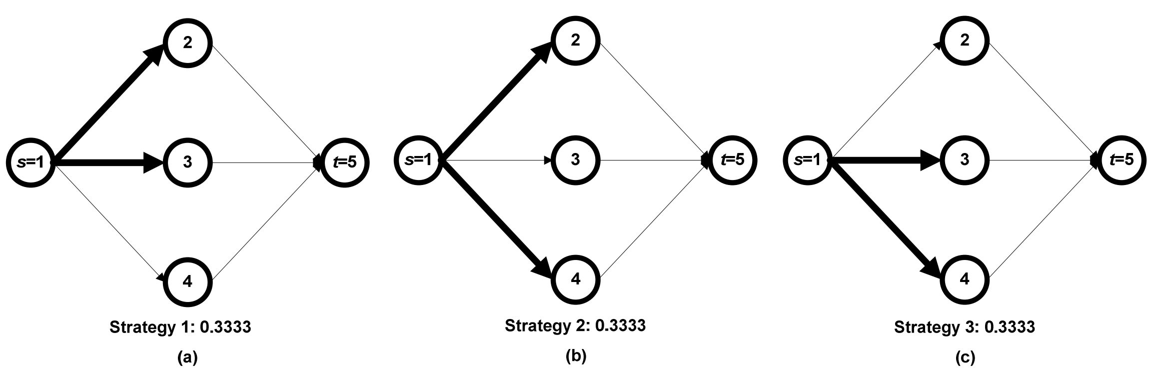

Figure 5. Solution to the problem in Figure 3 produced by marginal-probability procedure ($p_{e1} = 0.1 \,\forall e \in E$). The solution gives $E_D = P_I = 0.0667$.

We have already verified the correctness in finding expected number of detections for the marginal-probability model coupled with column generation for computing an implementable mixed strategy for the interdictor. In addition to that, when computing interdiction probability, the marginal-probability procedure works well and computes P_I correctly for the specific example with two inspectors for which the direct solution procedure fails to produce a correct value for P_I. Although both direct solution procedure and marginal-probability procedure produce correct results for expected number of detections, the former does not provide the correct solution for interdiction probability. The next chapter will test the two procedures on larger networks.

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V. COMPUTATIONAL RESULTS

This chapter investigates the solution of three sets of test problems. Computational tests are carried out on the first two problem sets for both the direct and the marginal-probability solution procedures. Because the marginal-probability procedure turns out to be substantially more efficient, only that procedure is applied to the last set of test problems, which involves large networks. All tests are carried out on a laptop computer with an Intel Core 2 Duo T6400 / 2 GHz (Dual-Core) processor, 4 gigabytes of RAM, the Microsoft Windows Vista Home Premium operating system, and with programs written and compiled using General Algebraic Modeling System (GAMS 2009) and with the CPLEX optimizer (GAMS-CPLEX 2009). “Solution times” reported here include all model-generation solution and report-writing time. Only default options are used with CPLEX.

A. TEST PROBLEMS

The test problems for this thesis cover (a) an “infiltration network” used to model the interdiction of an evader trying to enter illegally into the United States at the Arizona-Mexico border (see Appendix A), (b) a “Large Grid” network based on an 11×13 rectangular grid of vertices (this network has 145 vertices and 544 directed edges), and (c) a collection of “Larger Grid” networks with up to 2,000 vertices and 10,260 directed edges. All the edges have a detection probability known by both inspector and evader. Only the edges, $e \in E'$, i.e., those edges of the form (*s*,*v*) and (*v*,*t*) have zero probabilities. All the networks in those test problems are actually directed networks, but for simplicity are drawn as if they were undirected edges. Each “undirected edge,” except if incident to *s* or *t*, actually represents a pair of anti-parallel directed edges.

The test problems here vary in numbers of edges, number of vertices, number of inspectors, and edge detection probabilities. We will test both solution procedures and determine at what point each has “difficulties” with respect to correctly computing P_I, and with respect to computational efficiency. Also, by changing edge detection

probabilities, the solution procedures are tested in terms of number of iterations and solution times. Table 1 summarizes the test problems that will be used in this chapter.

Network	$\|E\|$	$\|V\|$	Instance Code	Num. of Inspectors (m_i)	Edge Detection Probabilities
Mexico-Arizona Infiltration	109	39	**M.1.E**	$m_1 = 1$	All equal
			M.1.D	$m_1 = 1$	All different
			M.3.D	$m_1 = 3$	All different
			M.2T.D	$m_1 = 1, m_2 = 1$	All different
			M.2T.V.D	$m_1 =$ various $m_2 =$ various	All different
			M.3T.V.D	$m_1 =$ various $m_2 =$ various $m_3 =$ various	All different
			M.10.E	$m_1 = 10$	All equal
			M.14.E	$m_1 = 14$	All equal
Large Grid	544	145	**L.1.D**	$m_1 = 1$	All different
			L.2T.V.D	$m_1 =$ various $m_2 =$ various	All different
Larger Grid	Up to 10260	Up to 2000		$m_1 =$ various $m_2 =$ various	All different

Table 1. Summary of test-problem statistics. The codes are used in the text to refer to particular problems.

B. RESULTS FOR TEST PROBLEM 1

Figure 6 illustrates the infiltration network near the Arizona-Mexico border. The network is based loosely on the network used in Pulat (2005) for testing a sequential-play network-interdiction model. An evader is trying to traverse from the source vertex in Mexico (vertex *s* at the bottom) to sink vertex in Phoenix, Arizona (vertex *t* at the top.) Each edge emanating from the source vertex has a probability of detection equaling zero because this area is not under the control of the American interdictor. The detection probability is zero on each edge that enters the sink vertex because once the evader reaches those edges, it becomes impossible to detect him. (The northernmost echelon of vertices lies on an interstate freeway, so once the evader reaches there, he is “home

free.") The interdictor has control over three inspector types. The first inspector type has detection probabilities $p_{e1} \in [0.01, 0.30]\ \forall e \in E \setminus E'$, the second inspector type has detection probabilities $p_{e2} \in [0.01, 0.15]\ \forall e \in E \setminus E'$, and the third inspector type has detection probabilities $p_{e3} \in [0.01, 0.25]\ \forall e \in E \setminus E'$. Each probability is randomly generated, using a uniform distribution, for each edge.

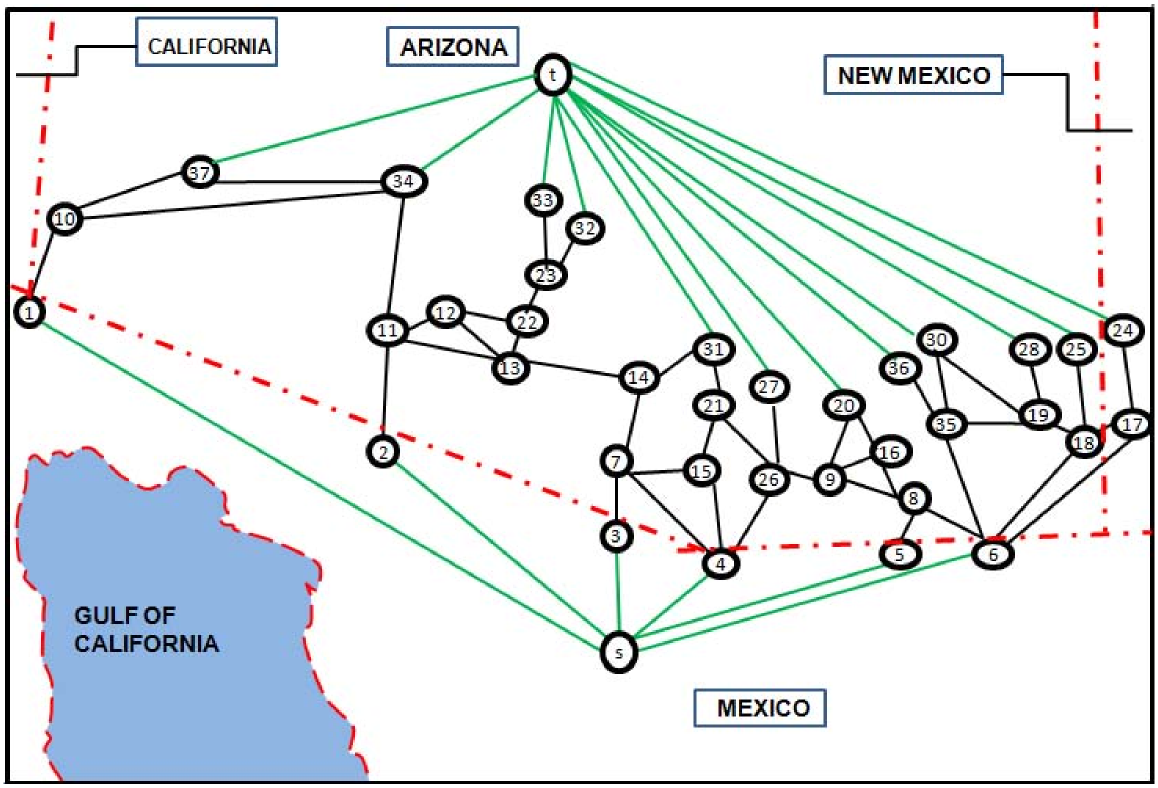

Figure 6. Infiltration network near the Arizona-Mexico border. The dashed straight lines indicate the borders between Arizona and Mexico, Arizona and New Mexico, and Arizona and California. Green (lighter) edges have a probability of detection of zero and black edges have detection probability greater than zero. The actual test network is directed and has 39 vertices and 109 directed edges; see the Appendix for details.

We will run some tests for different situations to (a) find the optimal expected number of detections, (b) determine if that matches interdiction probability, and (c) actually find the interdictor's optimal mixed inspector-assignment strategy if a valid one exists. The maximum solution time for any of the problems in this section is 47 seconds.

Because these times are so short, we do not compare solution times until we have a larger network to investigate, in the following section. We will use both of the solution procedures, starting with the direct solution procedure, then looking at the marginal-probability procedure, and finally comparing results.

In problem instance **M.1.E**, we assume that the interdictor has only one inspector of type 1 and none of any other types. Also, all detection probabilities are equal ($m_1 = 1$, and $p_{e1} = 0.1\ \forall e \in E \setminus E'$). The results from both procedures give exactly the same optimal mixed strategy for the interdictor with $E_D = P_I = 0.01$. The union of pure strategies forms a minimum-cardinality cut in the network, which has a cardinality of 10. (Actually, when speaking of "minimum cardinality cuts" we restrict consideration to those whose edges all have a probability of detection greater than 0.)

In problem instance **M.1.D**, we assume that the interdictor has only one inspector of a first inspector type and none from any other type. This time the detection probabilities are different, however: $p_{e1} \in [0.01, 0.30]\ \forall e \in E \setminus E'$.

The direct solution procedure produces the solution we expect. Table 2 shows the optimal mixed inspector assignment strategy.

a	x'_a	Edge Assignment for Inspector
1	0.1957	(2,11)
2	0.0799	(6,35)
3	0.0584	(9,20)
4	0.0955	(10,34)
5	0.0851	(10,37)
6	0.0778	(14,13)
7	0.0416	(14,31)
8	0.0638	(16,20)
9	0.0425	(17,24)
10	0.1039	(18,19)
11	0.0408	(18,25)
12	0.0515	(21,31)
13	0.0635	(26,27)

Table 2. Test M.1.D: Optimal mixed inspector-assignment strategy produced by the direct solution procedure when one inspector of type 1 is used and none of any other types ($p_{e1} \in [0.01, 0.30]\ \forall e \in E \setminus E'$). The solution gives $E_D = P_I = 0.0117$. Pure assignment strategies consist of assigning the interdictor to a single edge.

Figure 7 displays the optimal mixed strategy for the interdictor as produced by the direct solution procedure. In this case, the pure strategies involve the assignment of the inspector to one of 13 edges. The edges that have an inspector assigned to them form an *s*-*t* cut (although not a minimum-cardinality cut as in **M.1.E**).

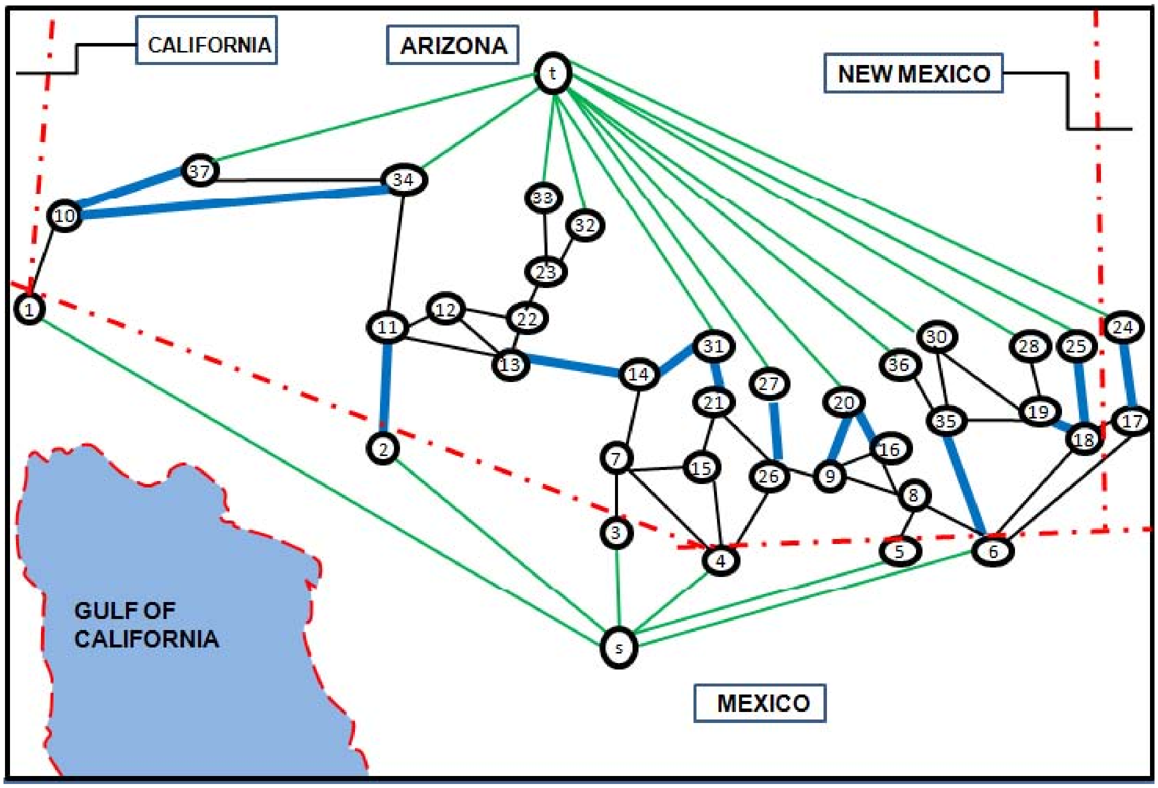

Figure 7. Test M.1.D: Pure inspector-to-edge assignment strategies found by the direct solution procedure, indicated by thick blue lines: a pure strategy consists of selecting a single edge in this test problem. The solution gives $E_D = P_I = 0.0117$.

The marginal-probability procedure produces identical results.

There might be an administrative advantage to assigning inspectors to the fewest edges possible. Therefore, we would like to determine P_I when the inspector is restricted to the minimum cardinality cut, and find out how much we lose in interdiction probability if we do that. Since the minimum cardinality cut for this network has 10 edges, and the strategies for this particular problem instance form a cut with 13 edges, we try to save our

effort to assign inspectors on fewer edges if the restricted solution is the same as the original one. Table 3 displays the optimal mixed strategy when the inspector is restricted to the edges in the minimum-cardinality cut.

a	x'_a	Edge Assignment for Inspector
1	0.3582	(1,10)
2	0.1075	(2,11)
3	0.0448	(4,26)
4	0.0303	(5,8)
5	0.0626	(6,8)
6	0.1535	(6,17)
7	0.0504	(6,18)
8	0.0439	(6,35)
9	0.1057	(7,14)
10	0.0433	(15,21)

Table 3. Test M.1.D: Optimal mixed strategy produced by the marginal-probability procedure when one inspector of type 1 is used and no others and the solution is restricted to the minimum cardinality cut ($p_{e1} \in [0.01, 0.30]\ \forall e \in E \setminus E'$). The solution gives $E_D = P_I = 0.0064$.

Figure 8 shows the unique minimum cardinality cut in the infiltration network. The overall interdiction probability becomes 0.0064 when the solution for the problem instance **M.1.D** is restricted on the minimum cardinality cut. Each thick blue line represents a pure inspector assignment for the interdictor.

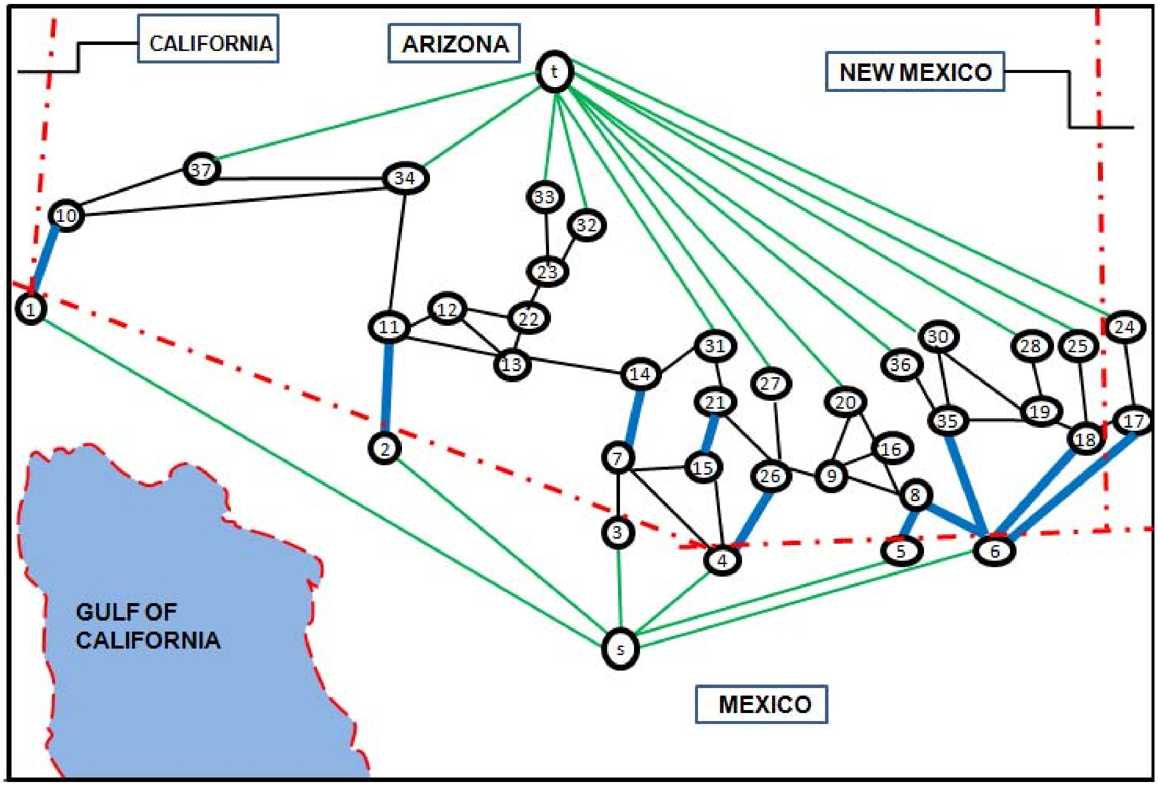

Figure 8. Test M.1.D: Union of pure inspector assignment strategies in the infiltration network when the solution is restricted to the minimum cardinality cut indicated by thick blue lines ($p_{e1} \in [0.01, 0.30]\ \forall e \in E \setminus E'$). The solution gives $E_D = P_I = 0.0064$.

If we do not restrict inspectors to the minimum-cardinality cut, we achieve $P_I = 0.0117$, which is better than the value of 0.0064 for the restricted solution. So, it is up to the interdictor to decide whether the administrative advantage of having the inspector work on fewer edges outweighs the loss in interdiction probability.

In problem instance **M.3.D**, we assume that we have three inspectors of type 1, none of any other types, and $p_{e1} \in [0.01, 0.30]\ \forall e \in E \setminus E'$. Table 4 shows the pure inspection strategies (those that have positive probability) produced by direct solution procedure:

		Inspector Assignments (Pure Strategies)												
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
Edges	(2,11)				1			1	1	1			1	1
	(6,35)			1						1				1
	(9,20)	1								1				
	(10,34)	1					1					1		
	(10,37)			1	1			1						
	(14,13)					1		1						
	(14,31)			1							1		1	
	(16,20)	1	1									1		
	(17,24)		1				1							
	(18,19)		1			1	1					1		
	(18,25)				1						1			1
	(21,31)					1			1				1	
	(26,27)								1		1			

Table 4. Test M.3.D: Inspector assignment strategies (a_i) for the interdictor produced by the direct solution procedure. There are three interdictors of type 1, none of any other types, and $p_{e1} \in [0.01, 0.30]\ \forall e \in E \setminus E'$. For instance, a_1 puts one inspector on edge (9,20), a second on edge (10,34), and a third on edge (16,20).

When we look at the pure inspector assignment strategies produced by the direct solution procedure, we see that the first edge is used in six different strategies. The union of the edges in the interdictor's pure strategies forms an *s*-*t* cut with 13 edges. When the very same problem is solved with the marginal-probability procedure, we obtain the same expected number of detection and interdiction probability, which is 0.0352. The marginal probabilities that both models produce are exactly the same. But the strategies differ because there are multiple optimal solutions to the problem. So, when we solve the same problem with the marginal-probability procedure, we find the 13 pure strategies as shown in Table 5.

		Inspector Assignments (Pure Strategies)												
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
Edges	(2,11)	1			1	1		1					1	
	(6,35)					1	1							
	(9,20)							1			1			1
	(10,34)	1					1			1				
	(10,37)				1							1		
	(14,13)			1	1									1
	(14,31)			1				1						
	(16,20)	1							1					
	(17,24)										1	1	1	
	(18,19)		1				1		1				1	
	(18,25)		1								1			
	(21,31)		1							1		1		1
	(26,27)			1		1			1	1				

Table 5. Test M.3.D: Inspector assignment strategies (a_i) for the interdictor produced by the marginal-probability procedure ($p_{e1} \in [0.01, 0.30]\ \forall e \in E \setminus E'$).

For this problem $E_D = P_I = 0.0352$. Let us again look to see if the overall interdiction probability will change if we decide to restrict our solution to the minimum cardinality cut. Table 6 shows the strategies produced on the minimum cardinality cut:

		Inspector Assignments (Pure Strategies)								
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
Edges	(1,10)	1	1	1	1	1	1	1	1	1
	(2,11)	1				1	1			
	(4,26)						1			
	(5,8)				1					
	(6,8)				1				1	1
	(6,17)		1			1		1		
	(6,18)	1						1		
	(6,35)			1					1	
	(7,14)		1							
	(15,21)			1						1

Table 6. Test M.3.D: Inspector assignment strategies (a_i) for the interdictor produced by the marginal-probability procedure on the minimum cardinality cut.

When we restrict the solution to the minimum cardinality cut, the overall interdiction probability drops from 0.0352 to 0.0180. Table 6 shows the corresponding strategies from the marginal-probability procedure.

In the next test, problem instance **M.2T.D**, we have one inspector of each of two types. We will again solve the problem with each of the two procedures, and then investigate what happens if we restrict the solutions to the minimum cardinality cut. Table 7 shows the probability of each pure assignment strategy for the interdictor:

Marginal-Probability Procedure					Direct Solution Procedure			
a	Edge Assignment for r=1	Edge Assignment for r=2	x'_a		a	Edge Assignment for r=1	Edge Assignment for r=2	x'_a
1	(16,20)	(2,11)	0.0746		1	(9,20)	(6,18)	0.0977
2	(17,18)	(6,18)	0.0618		2	(16,20)	(1,10)	0.1626
3	(17,24)	(6,35)	0.1084		3	(17,18)	(6,35)	0.0618
4	(18,19)	(7,14)	0.1619		4	(18,19)	(2,11)	0.1566
5	(18,25)	(1,10)	0.0615		5	(21,31)	(1,10)	0.0607
6	(9,20)	(6,35)	0.0788		6	(17,24)	(6,35)	0.1084
7	(26,27)	(1,10)	0.1617		7	(26,27)	(7,14)	0.1617
8	(18,25)	(2,11)	0.0020		8	(18,25)	(7,14)	0.0635
9	(16,20)	(6,18)	0.0359		9	(21,31)	(6,35)	0.0691
10	(9,20)	(7,14)	0.0701		10	(9,20)	(2,11)	0.0511
11	(21,31)	(2,11)	0.1312		11	(21,31)	(7,14)	0.0014
12	(16,20)	(6,35)	0.0522		12	(18,19)	(7,14)	0.0052

Table 7. Test M.2T.D: Mixed inspector assignment strategy for the interdictor produced by the marginal-probability and direct solution procedures when one inspector of each of two types is used ($p_{e1} \in [0.01, 0.30]$, $p_{e2} \in [0.01, 0.15]\ \forall e \in E \setminus E'$). The solution gives $E_D = P_I = 0.0299$.

Both procedures produce the same optimal objective value, $E_D = P_I = 0.0299$. The interdictor's pure inspector assignments put the inspectors on edges that form an *s*-*t* cut that has 12 edges for the marginal-probability procedure and 13 edges for the direct solution procedure. As was the case in the previous example, the strategies differ and that is again because there are multiple optimal solutions to the problem.

Let us now look at how much we lose if we restrict our solution to the minimum cardinality cut.

a	Edge Assignment for r=1	Edge Assignment for r=2	x'_a
1	(4,26)	(1,10)	0.0052
2	(6,8)	(6,18)	0.1483
3	(6,17)	(6,35)	0.1756
4	(6,17)	(1,10)	0.1599
5	(6,35)	(7,14)	0.0012
6	(6,17)	(2,11)	0.1536
7	(5,8)	(7,14)	0.1038
8	(4,26)	(15,21)	0.1484
9	(6,8)	(7,14)	0.0664
10	(6,17)	(6,18)	0.0376

Table 8. Test M.2T.D: Optimal mixed assignment strategy for the interdictor when one inspector of each of the two types is used ($p_{e1} \in [0.01, 0.30]$, $p_{e2} \in [0.01, 0.15]$ $\forall e \in E \setminus E'$). This solution gives $E_D = P_I = 0.0221$.

The overall interdiction probability for the original problem is 0.0299. When we restrict the solution to putting an inspector on edges of the minimum-cardinality cut, the interdiction probability drops to 0.0221; the number of pure strategies decreases from 12 to 10, however Again, the interdictor's would not determine if the loss of detection probability is worth the advantage of having fewer edges to deal with.

The problem instances, so far, have shown that both of the solution procedures produce same expected number of detections and interdiction probability for the interdictor in the infiltration network. Also the problems indicate that the number of detection decreases when the solution is restricted on the minimum cardinality cut. In the next test, on problem instance **M.2T.V.D**, we extend **M.2T.D** to different numbers of inspectors. Table 9 summarizes the results for both procedures.

m_1	m_2	$P_I = E_D$
1	1	0.0299
1	2	0.0424
2	1	0.0461
1	3	0.0537
3	1	0.0611
2	3	0.0728
3	2	0.0761
3	3	0.0898
5	5	0.1449

Table 9. Test M.2T.V.D: Optimal interdiction probabilities for different number of inspectors of each of two inspector types ($p_{e1} \in [0.01, 0.30]$, $p_{e2} \in [0.01, 0.15]$ $\forall e \in E \setminus E'$).

Next, in problem instance **M.3T.V.D**, we extend **M.2T.V.D** to different numbers of inspectors and types. Table 10 summarizes the problem variants and results. Both solution methods produced the same results.

m_1	m_2	m_3	$P_I = E_D$
1	1	1	0.0447
1	1	2	0.0576
1	2	3	0.0831
1	3	3	0.0958
2	3	3	0.1161
3	2	4	0.1322
3	3	3	0.1331
3	3	4	0.1420
8	1	1	0.1438
4	3	3	0.1466
5	2	3	0.1481

Table 10. Test M.3T.V.D: Various examples with different number of inspectors of each of three inspector type and expected number of detections associated with it. One can see that each interdiction probability matches the expected number of detections. ($p_{e1} \in [0.01, 0.30]$, $p_{e2} \in [0.01, 0.15]$ $p_{e3} \in [0.01, 0.25]$ $\forall e \in E \setminus E'$).

For all tests conducted thus far, both solution procedures produce solutions with $P_I = E_D$. The pure inspector assignment strategies provided by the solution procedures put the inspectors on edges that form an *s*-*t* cut in the network and also no more than one inspector has been assigned to more than one edge on a path that the evader might use.

In the next test, **M.10.E**, we set $p_{e1} = 0.1 \, \forall e \in E \setminus E'$ and use 10 inspectors of type 1 (and none from any other types). This number of inspectors equals the number of edges in the minimum cardinality cut, and we expect the union of pure strategies to identify just that cut. We run the problem with the direct solution procedure and get $E_D = 0.1$. The direct solution procedure produces six different pure strategies that are not contained entirely with the minimum cardinality cut. Additionally, some pure strategies place inspectors such that the evader has more than one detection opportunity on a given path. Thus, interdiction probability and expected number of detections will not be equal in this solution. In contrast, the marginal-probability procedure finds a solution with $P_I = E_D$.

It is also important to note that the solution time for the direct solution procedure is 44.7 seconds whereas the solution time for the marginal-probability procedure is only 0.5 seconds.

As a final example, in problem instance **M.14.E**, when both of the procedures are run with 14 inspectors of type 1 with $p_{e1} = 0.1 \, \forall e \in E \setminus E'$, we obtain $E_D = 0.1267$. Both solution procedures fail to give correct answers since the number of inspectors exceeds the cardinality of the minimum cardinality cut.

C. RESULTS FOR TEST PROBLEM 2

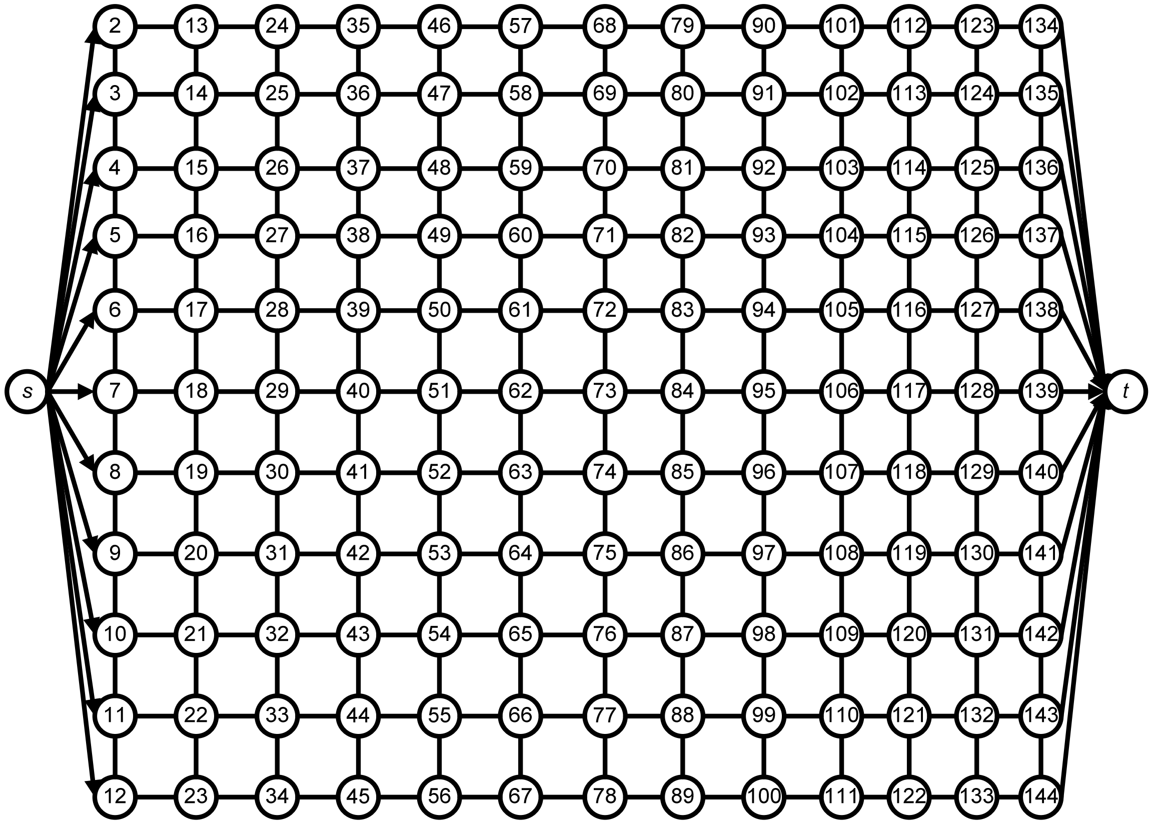

Figure 9. Large Grid network. The actual edges are directed, and the directed network has 145 vertices, including source and sink, and 544 edges. “*s*” is the source vertex or starting point and “*t*” is the sink vertex or destination point.

The directed version of the grid network in Figure 9 has 145 vertices and 544 edges. It is a large network and we have two inspector types with various values for m_r. We compute optimal interdiction probabilities for different numbers of inspectors of each type and present those values in Table 11; corresponding solution times for the marginal-probability solution procedure appear in Table 12, and for the direct solution procedure in Table 13. The interdiction probability increases for each additional inspector that we add. Solution times for the marginal-probability procedure are modest when compared to the solution times when the direct solution procedure is used. For example when six inspectors of each of two types are used, the marginal-probability procedure solves the network interdiction problem (in Figure 9) in about 15 seconds, whereas the direct solution procedure solves the same problem in about 26 minutes.

		$r=2$					
	m_r	1	2	3	4	5	6
$r=1$	1	0.0440	0.0664	0.0871	0.1073	0.1275	0.1476
	2	0.0636	0.0880	0.1110	0.1328	0.1541	0.1743
	3	0.0812	0.1091	0.1321	0.1550	0.1774	0.1982
	4	0.0985	0.1272	0.1531	0.1761	0.1979	0.2192
	5	0.1151	0.1449	0.1729	0.1961	0.2177	0.2392
	6	0.1317	0.1624	0.1906	0.2160	0.2376	0.2590

Table 11. Test L.2T.V.D: Optimal interdiction probabilities for different number of inspectors of each of two inspector types.

		$r=2$					
	m_r	1	2	3	4	5	6
$r=1$	1	6.2	7.5	7.8	8.8	7.7	12.5
	2	6.3	5.4	6.3	7.1	7.3	7.6
	3	5.8	5.9	5.9	6.4	9.7	10.2
	4	5.7	6.2	7.6	8.7	9.7	9.9
	5	5.8	7.1	7.8	8.8	9.7	10.2
	6	6.4	9.6	9.7	10.8	12.4	14.9

Table 12. Test L.2T.V.D: Solution time (seconds) for the marginal-probability procedure for varying numbers of inspectors of two types.

		$r=2$					
	m_r	1	2	3	4	5	6
$r=1$	1	423.4	469.6	555.9	557.6	618.1	703.7
	2	475.7	561.4	645.3	694.2	712.8	735.0
	3	564.5	662.3	745.6	779.0	872.7	964.6
	4	644.5	718.4	801.1	1016.4	1048.5	1255.8
	5	709.7	722.2	843.1	1052.6	1160.4	1424.6
	6	684.5	899.9	1079.9	1200.3	1451.6	1572.3

Table 13. Test L.2T.V.D: Solution time (seconds) for the direct solution procdure for varying number of inspectors of two types.

D. RESULTS FOR "LARGER GRID" NETWORKS

From the computations performed so far, we find that the direct solution procedure is substantially slower than the marginal-probability solution procedure, and does not always yield a solution in which expected number of detections equals interdiction probability. So, we will not investigate further computations using the direct solution procedure. We now explore the computational efficiency of the marginal-probability procedure on larger networks, and will specify a number of inspectors of two types that is not so large that the solution is in danger of being invalid. That is, we will not specify a total number of inspectors that exceeds the size of a minimum cardinality cut. We investigate grid networks that resemble the large grid network of the previous section.

Now, let us start by expanding our grid network in Figure 9 and test the computational efficiency of marginal-probability procedure in various larger grid networks. In these tests we assume that $p_{e1} \in [0.10, 0.40], \forall e \in E \setminus E'$, and $p_{e2} \in [0.30, 0.60], \forall e \in E \setminus E'$.

m_1	m_2	P_I	Num. of Iterations	Soln. Time (sec.)
1	1	0.0415	1021	54
1	2	0.0667	1019	50
2	2	0.0830	1061	48
3	4	0.1502	1075	54
4	4	0.1659	1029	50
3	5	0.1752	1034	54
5	5	0.2112	1051	57
6	6	0.2489	1072	51
7	6	0.2645	1035	58
8	8	0.3318	1097	75
8	9	0.3574	1070	81
10	10	0.4143	1097	79

Table 14. Computational efficiency of the marginal-probability procedure as a function of the number of inspectors in a Large Grid network with 402 vertices (with a rectangular grid (height × length) = (20 × 20)).

Table 14 shows that the number of iterations and solution time increase only moderately as the number of inspectors increases.

p_{e1}	p_{e2}	P_I	Num. of Iterations	Soln. Time (sec.)
[0.01,0.02]	[0.01,0.02]	0.0086	1377	89
[0.01,0.02]	[0.90,0.95]	0.2376	1292	83
[0.15,0.20]	[0.05,0.15]	0.0801	1195	83
[0.15,0.20]	[0.35,0.45]	0.1689	1253	69
[0.45,0.50]	[0.90,0.95]	0.3564	857	64
[0.45,0.50]	[0.45,0.50]	0.2437	815	51
[0.90,0.95]	[0.90,0.95]	0.4688	816	51

Table 15. Computational efficiency of the marginal-probability procedure as a function of detection probabilities with $m_1 = m_2 = 5$ on the same 20×20 grid network as in Table 14.

In Table 15, we change the detection probabilities for one case from Table 14 and that is: the interdictor has five inspectors of each of the two types. For instance when $p_{e1} \in [0.01, 0.02]$, $p_{e2} \in [0.90, 0.95]$, $\forall e \in E \setminus E'$ the interdiction probability becomes, $P_I = E_D = 0.2376$, and the number of iterations to obtain that solution becomes 1292. On the other hand when we have $p_{e1} \in [0.45, 0.50]$, $p_{e2} \in [0.90, 0.95]$, $\forall e \in E \setminus E'$, we find $P_I = E_D = 0.3564$, and the number of iterations to reach that solution is 857. So, it can be seen that when we decrease detection probability for the edges, it not only affects the interdiction probability but also increases notably the number of iterations.

Network Size (Height × Width)	L/M/H	m_1	m_2	P_I	Num. of Iterations	Soln. Time (sec.)
15 × 20	Low	1	1	0.0705	740	47
15 × 20	Med.	3	3	0.2114	830	46
15 × 20	High	6	6	0.4229	950	54
25 × 30	Low	2	2	0.0703	2414	131
25 × 30	Med.	6	6	0.2111	2771	143
25 × 30	High	10	10	0.3516	2997	181
30 × 30	Low	3	3	0.1055	2748	156
30 × 30	Med.	9	9	0.3164	2824	170
30 × 30	High	12	12	0.4221	3241	187
30 × 40	Low	7	7	0.2461	3672	1341
40 × 50	Med.	9	12	0.2502	6665	2355

Table 16. Computational efficiency of the marginal-probability procedure as grid-network size increases. The number of inspectors is "low," "medium" and "high" for each network size (but the total number for "high" is still less than a minimum-cardinality cut).

In Table 16, we have $p_{e1} \in [0.80, 0.90]$, $p_{e2} \in [0.10, 0.20]$, $\forall e \in E \setminus E'$. When the network size is increased, the solution time and number of iterations changes depending on the number of inspectors. Specifically as the network size grows large, it becomes harder for the marginal-probability procedure to solve the problem, and solution time and number of iterations grow at a super linear rate.

E. CONCLUSIONS ON COMPUTATIONAL TESTS

Computational testing indicates that

1. The direct solution procedure finds a mixed strategy for the interdictor in a single phase, and always computes the expected number of detections correctly. But the expected number of detections does not always reflect the interdiction probability in the network. The pure strategies produced by the direct solution procedure sometimes place the inspectors on more than one edge on a path that the evader might use, even when the number of inspectors used is less than cardinality of the minimum cardinality cut. That results in a difference between the E_D and P_I . Although this difference may not be large when the detection probabilities are small, it

may be large if the detection probabilities are large. We conclude that the direct solution procedure sometimes yields correct solutions, but also can give incorrect solutions, and is much slower than the marginal-probability procedure. Thus, the marginal-probability procedure is preferred.

2. For the marginal-probability procedure, we find that an optimal solution is always produced unless the number of inspectors exceeds the minimum cardinality cut. It is notably faster than the direct solution procedure, but solution times grow super linearly as the network size gets larger.

    a. The marginal-probability procedure is a two-phase procedure which first needs a set of marginal probabilities for interdictor's edge inspection and then uses those probabilities to find a mixed inspector-assignment strategy for the interdictor.

    b. The marginal-probability procedure computes the expected number of detections like the direct solution procedure does. But, if the number of inspectors does not exceed the cardinality of a minimum cardinality cut, the procedure always gives an answer in which expected number of detections equals interdiction probability..

    c. The various examples in Figure 9 on the large grid network indicated that the marginal-probability procedure is better than the direct solution procedure in terms of solution time. But the solution time and number of iterations in other larger networks such as a network with 2,000 vertices and 5,000 edges, increase and become large.

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VI. CONCLUSIONS

This thesis has developed two different solution procedures for a model in which an "interdictor," who controls multiple inspectors of different types, seeks to detect an "evader" that is traveling through a directed network from a known source vertex *s*, to a known sink vertex *t*. The model is formulated and solved as a two-person, zero-sum, simultaneous-play, network-interdiction game. The objective for the interdictor is to find a probabilistic (mixed) inspector-to-edge assignment strategy that maximizes "interdiction probability," i.e., the probability that the evader is detected while trying to traverse the network. The evader seeks to minimize interdiction probability through a probabilistic path-selection strategy.

Several simplifying assumptions are made: (a) an inspector assigned to an edge (i, j) cannot detect an evader that is travelling on edge (j,i), (b) only one inspector can be assigned to each edge, (c) there is one origin and one destination for the evader and they are known by the interdictor, and (d) the detection probability associated with each edge is known by both of the players. The basic model also uses a surrogate objective function that evaluates expected number of detections. One goal of the thesis is to determine when a solution that maximizes that objective also maximizes interdiction probability.

Two different solution procedures are presented to find an optimal mixed inspector-assignment strategy. (The evader's optimal strategy is not particularly important to us.) The first "direct solution procedure," finds an optimal solution in a single phase using a column-generation algorithm. The second "marginal-probability (solution) procedure" has two phases. The first-phase computes a marginal probability distribution for inspector-to-edge assignments in a relaxed linear-programming model; that probability distribution may or may not be valid. In a second column-generation phase, a complete, mixed strategy (i.e., joint distribution) is found with a marginal distribution that matches that found in the first phase, if possible. The linear program of the first phase is straightforward, and the column-generation problem in the second phase is simpler than the column-generation problem of the direct solution procedure.

The objective function of our network-interdiction model measures expected number of detections for simplicity. However, we would like that objective to measure the probability of detecting the evader, i.e., interdiction probability. Computational results indicate that, under reasonable restrictions on the data—the total number of inspectors should not exceed the cardinality of a minimum-cardinality cut—both solution procedures will solve correctly for expected number of detections. Furthermore, given those same restrictions, the expected number of detections can equal interdiction probability. However, even given those restrictions, the marginal probability procedure may be successful in finding a solution that optimizes both objectives when the direct procedure does not. The opposite situation never occurs.

A variety of computational tests indicate that the marginal-probability procedure is much faster than the direct procedure. For instance, for a grid network with 145 vertices, 544 edges, and 12 inspectors split evenly between two types, the marginal-probability procedure yields a solution in 15 seconds whereas direct solution procedure requires almost a half hour to solve the problem. Given that the marginal-probability solution procedure more often finds a solution that computes interdiction probability properly, and it is much faster, it is clearly preferred over the direct solution procedure.

A. RECOMMENDATIONS FOR FUTURE RESEARCH

This thesis has developed several models and algorithms without formal proofs, and without a discussion of theoretical computational complexity. These omissions should be corrected.

This thesis concerns itself only with network-interdiction problems involving a single evader. Further research may wish to address the problem of multiple evaders operating in a coordinated fashion; for example, see Hespanha et al. (1999), and Vidal et al. (2002).

The interdiction assets, i.e., different types of inspectors, are assumed in this thesis to have the same detection probability on a given edge at all times and under all weather conditions. That may not be true in reality, and the topic of variable detection probabilities warrants further study.

Extensions of the model in this thesis should also be considered (a) to account for uncertainty about the evader's origin and destination (see Pan et al. 2003), (b) to model an inspector who, when assigned to a edge (i, j), can detect an evader traveling on (j, i) as well as on (i, j), and (c) to handle situations in which the number of inspectors is large compared to the cardinality of a minimum-cardinality cut in the network. Note that issue (b) relates to modeling an undirected network.

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APPENDIX. INPUT DATA FOR INFILTRATION NETWORK

This table specifies data for the “infiltration network” tested in Chapter V.

from	to	r=1	r=2	r=3	from	to	r=1	r=2	r=3	from	to	r=1	r=2	r=3
s	1	0.000	0.000	0.000	12	11	0.192	0.104	0.176	23	22	0.209	0.102	0.201
s	2	0.000	0.000	0.000	12	13	0.254	0.096	0.176	23	32	0.256	0.027	0.256
s	3	0.000	0.000	0.000	12	22	0.082	0.065	0.098	23	33	0.231	0.022	0.265
s	4	0.000	0.000	0.000	13	11	0.214	0.137	0.123	24	17	0.260	0.044	0.159
s	5	0.000	0.000	0.000	13	12	0.254	0.096	0.176	24	t	0.000	0.000	0.000
s	6	0.000	0.000	0.000	13	14	0.151	0.035	0.131	25	18	0.288	0.053	0.178
1	10	0.018	0.134	0.122	13	22	0.243	0.114	0.199	25	t	0.000	0.000	0.000
2	11	0.060	0.144	0.111	14	7	0.061	0.129	0.122	26	4	0.144	0.064	0.144
3	7	0.240	0.025	0.145	14	13	0.151	0.035	0.131	26	9	0.106	0.067	0.167
4	7	0.290	0.150	0.190	14	31	0.282	0.132	0.199	26	21	0.162	0.058	0.111
4	15	0.219	0.113	0.156	15	4	0.219	0.113	0.156	26	27	0.185	0.119	0.119
4	26	0.144	0.064	0.144	15	7	0.189	0.087	0.088	27	26	0.185	0.119	0.119
5	8	0.213	0.109	0.109	15	21	0.149	0.149	0.148	27	t	0.000	0.000	0.000
6	8	0.103	0.075	0.089	16	8	0.123	0.112	0.111	28	19	0.147	0.061	0.132
6	17	0.042	0.028	0.049	16	9	0.017	0.053	0.098	28	t	0.000	0.000	0.000
6	18	0.128	0.119	0.111	16	20	0.184	0.106	0.185	30	19	0.146	0.061	0.128
6	35	0.147	0.125	0.127	17	6	0.042	0.028	0.049	30	t	0.000	0.000	0.000
7	3	0.240	0.025	0.145	17	18	0.188	0.097	0.098	31	14	0.282	0.132	0.199
7	4	0.290	0.150	0.190	17	24	0.276	0.044	0.159	31	21	0.228	0.042	0.231
7	14	0.061	0.129	0.122	18	6	0.128	0.119	0.111	31	t	0.000	0.000	0.000
7	15	0.189	0.087	0.088	18	17	0.188	0.097	0.098	32	23	0.256	0.027	0.256
8	5	0.213	0.109	0.109	18	19	0.113	0.043	0.034	32	t	0.000	0.000	0.000
8	6	0.103	0.075	0.089	18	25	0.288	0.053	0.178	33	23	0.231	0.022	0.265
8	9	0.253	0.133	0.111	19	18	0.113	0.043	0.034	33	t	0.000	0.000	0.000
8	16	0.123	0.112	0.111	19	28	0.147	0.061	0.132	34	10	0.123	0.098	0.132
9	8	0.253	0.133	0.111	19	30	0.146	0.061	0.128	34	11	0.007	0.042	0.017
9	16	0.017	0.053	0.098	19	35	0.040	0.000	0.043	34	37	0.244	0.150	0.223
9	20	0.201	0.098	0.053	20	9	0.201	0.098	0.053	34	t	0.000	0.000	0.000
9	26	0.106	0.067	0.167	20	16	0.184	0.106	0.185	35	6	0.147	0.125	0.127
10	1	0.018	0.134	0.122	20	t	0.000	0.000	0.000	35	19	0.040	0.000	0.043
10	34	0.123	0.098	0.132	21	15	0.149	0.149	0.148	35	30	0.204	0.050	0.109
10	37	0.138	0.143	0.151	21	16	0.263	0.041	0.246	35	36	0.061	0.045	0.054
11	2	0.060	0.144	0.111	21	31	0.228	0.042	0.231	36	35	0.061	0.045	0.054
11	12	0.192	0.104	0.176	22	12	0.082	0.065	0.098	36	t	0.000	0.000	0.000
11	13	0.214	0.137	0.123	22	13	0.243	0.114	0.199	37	10	0.138	0.143	0.151
11	34	0.007	0.042	0.017	22	23	0.209	0.102	0.201	37	34	0.244	0.150	0.223
										37	t	0.000	0.000	0.000

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